

Variants of the Kakeya problem over an algebraically closed field*

Kaloyan Slavov

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Abstract

First, we study constructible subsets of \mathbb{A}_k^n which contain a line in any direction. We classify the smallest such subsets in \mathbb{A}^3 of the type $R \cup \{g \neq 0\}$, where $g \in k[x_1, \dots, x_n]$ is irreducible of degree d , and $R \subset V(g)$ is closed. Next, we study subvarieties $X \subset \mathbb{A}^N$ for which the set of directions of lines contained in X has the maximal possible dimension. These are variants of the Kakeya problem in an algebraic geometry context.

1 Introduction

In [6], T. Wolff proposed a finite field model for the classical harmonic analysis Kakeya problem. Namely, he defines a Kakeya subset E of \mathbb{F}_q^n to be a subset which contains a line in any direction, in analogy with the notion of a Kakeya subset of \mathbb{R}^n , which is a compact subset containing a unit line segment in any direction. The finite field Kakeya problem was solved by Z. Dvir in [2] and has proved to be a useful model for the hard classical Euclidean problem. Recently in [1], Dummit and Hablicsek answered a question of Ellenberg, Oberlin, and Tao about Kakeya subsets of $\mathbb{F}_q[[t]]^n$; this is a version of the Kakeya problem over non-archimedean local rings.

We give two different *algebraic-geometry* versions of the Kakeya problem that are interesting over an algebraically closed field k , of any characteristic. Our main motivation is that the smallest known example of a Kakeya subset of \mathbb{F}_q^n arises from a Kakeya variety as in Definition 4. Thus, this extra structure of an algebraic variety coming with the smallest known example of a Kakeya subset of \mathbb{F}_q^n should not be neglected, and studying it presents sufficient motivation and independent interest. At the very least, this leads to interesting algebraic geometry questions and structural results. More importantly, algebraic geometry models tie with the general philosophy and metatheorem that extreme combinatorial configurations possess algebraic structure. The classical harmonic analysis Kakeya problem is

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notoriously difficult; on the other hand, the algebraic geometry version that we present here is approachable due to the rich structure coming with the hypothesis of constructibility.

In Section 2, we study a constructible subset E of \mathbb{A}^n which contains a line in any direction. In analogy with classical frameworks, we call such an E a *Kekeya subset* of \mathbb{A}^n . The starting point of our investigation in Section 2.1 is

Proposition 1. *a) If $E \subset \mathbb{A}_k^n$ is a constructible Kekeya subset, then $\dim E = n$.
b) Let $E \subset \mathbb{A}_k^n$ be an open subset. Then E is a Kekeya subset if and only if $\dim E^c \leq n-2$.*

Consider a constructible Kekey subset $E \subset \mathbb{A}_k^n$, where E is a finite disjoint union $E = \cup T_i$ of locally closed subsets T_i of \mathbb{A}_k^n . Each T_i is an open subset of a closed subset \hat{T}_i of \mathbb{A}^n , and for some i (uniquely determined), $U := T_i$ is open in \mathbb{A}^n by Proposition 1a. So, $E = R_1 \cup \dots \cup R_s \cup U$, where each R_i is locally closed in \mathbb{A}^n with $\dim \overline{R_i} \leq n-1$, and U is a nonempty open in \mathbb{A}^n .

If $\dim U^c \leq n-2$, then U itself is a Kekeya set by Proposition 1b.

Consider now the “small” case when $\dim U^c = n-1$. So, U^c is a union of finitely many irreducible hypersurfaces, together with some lower-dimensional irreducible components. We say that E is of type t if U^c has exactly t irreducible components, all of them of dimension $n-1$. The larger the t , the smaller the U . We focus on the case $t=1$ and consider constructible Kekeya subsets of \mathbb{A}^n of type 1; our goal is to describe the smallest such subsets.

We now study the decomposition

$$E = R_1 \cup \dots \cup R_s \cup \{g \neq 0\} \quad (1)$$

(where $g \in k[x_1, \dots, x_n]$ is irreducible) of a Kekeya subset of type 1, in terms of the degree d of g . A larger degree d will correspond to a smaller Kekeya set. Further, for a fixed d , a small Kekeya set will have $R_1 \cup \dots \cup R_s$ small (the first measure will be its dimension).

In Section 2.2, we give an extreme example of a constructible Kekeya subset of \mathbb{A}^3 of type 1:

Proposition 2. *For any $d \geq 3$, there exists an irreducible hypersurface $V(g) \subset \mathbb{A}_k^3$ of degree d , and 2 points R_1, R_2 on it, such that $\{R_1, R_2\} \cup \{g \neq 0\}$ is a Kekeya subset of \mathbb{A}_k^3 .*

The main result of Section 2.3 is that the construction in the proof of Proposition 2 is essentially the only example of such an extreme small Kekeya subset of \mathbb{A}^3 of type 1; to state it precisely, we introduce some notation. For a subvariety $X \subset \mathbb{A}^n$, we denote by \overline{X} the projective closure of X in \mathbb{P}^n . The hyperplane at infinity in \mathbb{P}^n is denoted by $V(x_0)$, so x_1, \dots, x_n will be affine coordinates in \mathbb{A}^n , while $[x_0 : x_1 : \dots : x_n]$ will be projective coordinates in \mathbb{P}^n . For a polynomial $g \in k[x_1, \dots, x_n]$, we denote by G its homogenization with respect to x_0 , so $\overline{V(g)} = V(G)$.

Proposition 3. *Let $E = \{R_1, \dots, R_s\} \cup \{g \neq 0\} \subset \mathbb{A}_k^3$ be a Kekeya subset, where g is irreducible of degree $d \geq 3$, and $R_1, \dots, R_s \in V(g)$. Let $C = V(G) \cap V(x_0)$. Then C is*

irreducible, and $V(G)$ is a cone over it. Moreover, if C is non-flexy and non-funny¹, then C has a unique singular point, whose multiplicity is $d - 1$.

In particular, given an irreducible hypersurface $V(g) \subset \mathbb{A}^3$, unless g satisfies the very stingy requirements described in the statement above, it is not possible to add finitely many points R_1, \dots, R_s to $V(g)$ so that $\{R_1, \dots, R_s\} \cup \{g \neq 0\}$ is a Keakeya subset.

Next, in Section 3, we consider a subvariety $X \subset \mathbb{A}^N$ with $\dim X = n$. Let $V(x_0) = \mathbb{P}^N - \mathbb{A}^N$ be the hyperplane at infinity, and let $\Delta \subset V(x_0)$ be the set of all directions of lines contained in X . We say that X is a Keakeya subvariety of \mathbb{A}^N if the inequality $\dim \Delta \leq n - 1$ is an equality. As an example, we prove that any hypersurface $X \subset \mathbb{A}^N$ of degree $d < N$ is Keakeya in this sense. Next, we propose the following

Definition 4. A subvariety $X \subset \mathbb{A}_k^N$ of dimension n , together with a morphism $\pi : X \rightarrow \mathbb{A}_{z_1, \dots, z_n}^n$ is called a Keakeya cover if there exists an open subset $U \subset V(z_0) = \mathbb{P}_{[z_0, \dots, z_n]}^n - \mathbb{A}_{z_1, \dots, z_n}^n$ with the following property: for any $v \in U$, there exists a line $l \subset X \subset \mathbb{A}_k^N$ whose image under π is a line in \mathbb{A}^n in direction v .

The smallest known example of a combinatorial Keakeya subset of \mathbb{F}_q^n comes from

$$\{(a_1, \dots, a_{n-1}, b) \in \mathbb{F}_q^n \mid a_i + b^2 \text{ is a square in } \mathbb{F}_q \text{ for all } i\} \subset \mathbb{F}_q^n$$

(say q is odd, for convenience); see Appendix A in [4]. This is the image on \mathbb{F}_q -points of the composition

$$X = V(a_1 + b^2 - c_1^2, \dots, a_{n-1} + b^2 - c_{n-1}^2) \hookrightarrow \mathbb{A}_{a_1, \dots, a_{n-1}, b, c_1, \dots, c_{n-1}}^{2n-1} \rightarrow \mathbb{A}_{a_1, \dots, a_{n-1}, b}^n$$

which is a Keakeya cover as in Definition 4. Indeed, take $U = \{b \neq 0\}$ and for any direction $v = [\alpha_1 : \dots : \alpha_{n-1} : 1] \in U$, consider the line $a_i = \alpha_i t + \frac{\alpha_i^2}{4}, b = t, c_i = t + \frac{\alpha_i}{2}$. It is contained in X , and its image under $X \rightarrow \mathbb{A}^n$ is a line in direction v . This justifies the significance of Definition 4.

It is easy to prove the following

Proposition 5. Let $X \subset \mathbb{A}^N$ be an irreducible n -dimensional Keakeya subvariety of \mathbb{A}^N , and let d be the degree of its closure $\overline{X} \subset \mathbb{P}^N$. After performing $\text{codim} X$ appropriate linear projections, we obtain a finite map $\pi : X \rightarrow \mathbb{A}_{z_1, \dots, z_n}^n$ of degree d , which is a Keakeya cover.

In [5], we study covers which satisfy a more restrictive version of Definition 4.

Sections 2 and 3 are independent of one another and present different viewpoints towards a Keakeya problem in the context of algebraic geometry. Throughout the article, k is a fixed algebraically closed field.

¹See Section 2.3 for the definitions.

2 Constructible Takeya subsets

2.1 The geometry of constructible Takeya subsets

Proof of Proposition 1a. Suppose that $\dim E \leq n - 1$. Replacing E by its closure, we can assume without loss of generality that $E \subset \mathbb{A}_k^n$ is a closed subset. Let $V(x_0) = \mathbb{P}^n - \mathbb{A}^n$ be the hyperplane at infinity; the direction of a line in \mathbb{A}^n is a point in $V(x_0)$. Let $v \in V(x_0)$ be arbitrary. We know that there exists some line l in \mathbb{A}_k^n which is contained in E and whose projective closure \bar{l} passes through v . Taking closures in \mathbb{P}_k^n , the inclusion $l \subset E$ implies $\bar{l} \subset \bar{E}$ hence $v \in \bar{E}$. Since v was arbitrary, we deduce that $V(x_0) \subset \bar{E}$. However, since $\dim \bar{E} \leq n - 1$, this is possible only if $\dim \bar{E} = n - 1$, and $V(x_0)$ is one of its irreducible components. This is impossible, since $E \subset \mathbb{A}_k^n$, and hence cannot be a dense subset of $V(x_0)$. \square

Remark 6. We can compare and parallel the above proof with Dvir's proof of the Takeya problem in the finite field setting. Here \bar{E} plays the role of the hypersurface in Dvir's proof. The requirement $\deg f = d < q$ was needed in Dvir's work to derive a contradiction from $V(x_0) \subset V(\bar{f})$. Here in the geometrical setting, this is automatic. So, our proof of Proposition 1a is a geometric version of Dvir's argument.

Proof of Proposition 1b. Let $Z = \mathbb{A}_k^n - E$. Suppose first that $\dim Z \leq n - 2$. To show that E is a Takeya subset, consider an arbitrary $v \in V(x_0)$. Let \mathbb{G}_v be the subset of the Grassmanian $\mathbb{G}(1, n)$ consisting of all lines in \mathbb{P}^n passing through v . Consider the incidence correspondence

$$I := \{(x, l) \in Z \times \mathbb{G}_v \mid x \in l\} \subset Z \times \mathbb{G}_v,$$

together with its two projections to Z and \mathbb{G}_v . Each fiber of $I \rightarrow Z$ is 0-dimensional, so $\dim I = \dim Z \leq n - 2$. Since $\dim \mathbb{G}_v = n - 1$, there is a dense open subset of \mathbb{G}_v where every point is outside of the image of $I \rightarrow \mathbb{G}_v$. Any such line will be entirely contained in E .

Conversely, suppose that $\dim Z = n - 1$. We claim that E is not a Takeya subset. Let \bar{Z} be the closure of Z in \mathbb{P}^n . Note that $V(x_0) \not\subset \bar{Z}$. Let $v \in V(x_0) - \bar{Z}$. If $\bar{l} = l \cup \{v\} \in \mathbb{G}_v$, then \bar{l} must intersect \bar{Z} at some point w , by Bezout's theorem. Since $w \neq v$, we have $w \in \bar{Z} \cap \mathbb{A}^n = Z$, and so l is not contained in E . \square

2.2 Examples of small Takeya subsets of type 1

When $d = 1$, so $g = L$ is linear, the smallest such Takeya set would have to be $\{q\} \cup \{L \neq 0\}$ where q is a point on the hyperplane $V(L)$. In fact, for any $q \in V(L)$, the subset $E = \{q\} \cup \{L \neq 0\}$ is a Takeya subset of \mathbb{A}^n .

Proposition 7. *For any $d \geq 1$, there exists a Takeya set $E \subset \mathbb{A}_k^n$ of the form $E = R \cup \{g \neq 0\}$, where $g \in k[x_1, \dots, x_n]$ is irreducible of degree d , and R is a closed subset of $V(g)$ of dimension $n - 2$.*

Proof. Consider a smooth irreducible hypersurface $C \subset V(x_0) \simeq \mathbb{P}^{n-1}$ of degree d , and let $X = V(G) \subset \mathbb{P}^n$ be the projective cone over it, with vertex $v = [1 : 0 : \dots : 0]$. Choose a point $v' \in \mathbb{P}^n - X - V(x_0)$ and let X' be the cone over C but with vertex v' . Let $R = X \cap X'$; it is an $(n-2)$ -dimensional closed subvariety of X . Then $E = (\{v\} \cup R \cup \{G \neq 0\}) \cap \mathbb{A}^n \subset \mathbb{A}_k^n$ is a Kakeya subset of \mathbb{A}_k^n . \square

Lemma 8. *Let $X \subset \mathbb{P}_k^3$ be the cone over some irreducible curve $C \subset V(x_0)$, with vertex $v = [1 : 0 : 0 : 0]$. Suppose that C is not a line (equivalently, that X is not a plane). If a line $l \subset \mathbb{P}^3$ is contained in X , then the vertex v of X belongs to l .*

Proof. Consider the projection map $\mathbb{P}^3 - \{v\} \rightarrow V(x_0)$. If l does not pass through v , then its image is well-defined and is a line l' in $V(x_0) \simeq \mathbb{P}^2$. By the definition of a cone as the union of all lines connecting points of C with v , we must have $l' \subset C$. By the irreducibility of C , this would yield $l' = C$, which is a contradiction to the hypothesis. \square

When $d = 2$, so $g = Q$ is irreducible quadratic, again a Kakeya type-1 set $E = R_1 \cup \dots \cup R_s \cup \{Q \neq 0\}$ in \mathbb{A}^3 would have to contain at least one point of $V(Q)$. In fact, adding just one point suffices, if Q is chosen appropriately: we now give an example of such a Kakeya set $E = \{q\} \cup \{Q \neq 0\}$. Let C be a smooth quadric hypersurface in $V(x_0) \simeq \mathbb{P}^2$ defined by $Q \in k[x_1, \dots, x_3]$, and let $V(Q) \subset \mathbb{P}^3$ be the cone over C , with vertex at $v = [1 : 0 : 0 : 0]$. Then $E = \{v\} \cup \{Q \neq 0\}$ is a Kakeya subset of \mathbb{A}^3 .

One naturally asks for a refinement of this example. For a given d , we want to find an irreducible hypersurface $V(g) \subset \mathbb{A}_k^3$ of degree d and a subset $R \subset V(g)$ of dimension as small as possible, so that $R \cup \{g \neq 0\}$ is a Kakeya subset of \mathbb{A}_k^3 . Indeed, an extreme type-1 Kakeya subset of \mathbb{A}^3 as described in Proposition 2 exists, for any $d \geq 3$.

Proof of Proposition 2. Consider $C = V(y^{d-1}z - x^d) \subset \mathbb{P}^2 \simeq V(x_0)$, and let $X = V(y^{d-1}z - x^d) \subset \mathbb{P}_{[x_0:x:y:z]}^3$ be the cone over it. Let $R_1 = [1 : 0 : 0 : 0]$ be the vertex of the cone, and let $R_2 = [1 : 0 : 0 : 1]$ be a point which projects onto the singular point of C . Note that the multiplicity of R_1 is d and the multiplicity of R_2 is $d-1$.

Consider any $x \in V(x_0)$. If $x \notin C$, the line joining x and R_1 does not intersect $V(G)$ besides at R_1 . If $x \in C$ is a smooth point, the line joining x and R_2 does not hit $V(G)$ again. Finally, if $x = [0 : 0 : 0 : 1]$ is the singular point of C , then the line joining x and, say, $[1 : 1 : 0 : 1]$ does not intersect $V(G)$ again (besides at x). \square

2.3 Uniqueness of the cone construction

For a polynomial $g \in k[x_1, \dots, x_n]$ and a point $p \in \mathbb{A}_k^n$, we say that g vanishes at p with multiplicity m if the polynomial $g(x+p)$ has no terms of total degree less than m but has some nonzero terms of degree precisely m . For a line $l = \{p + tv \mid t \in k\}$ passing through p , recall that the intersection multiplicity $I_p(l, V(g))$ equals the order of vanishing at $t = 0$ of $g(p + tv) \in k[t]$.

Lemma 9. *Let $g \in k[x_1, \dots, x_n]$ be a polynomial of degree d and let $p \in V(g) \subset \mathbb{A}_k^n$. Let $\mathbb{G}_p \simeq \mathbb{P}^{n-1}$ be the variety of lines in \mathbb{A}_k^n (or, equivalently, in $\mathbb{P}_k^n = \mathbb{A}_k^n \cup V(x_0)$) passing*

through p . Suppose that there exists a dense subset $L \subset \mathbb{G}_p$ such that for each $l \in L$, we have $I_p(l, V(g)) = d$. Then the multiplicity of the point $p \in V(g)$ is exactly d .

Proof. Without loss of generality, $p = (0, \dots, 0)$. Write $g = g_1 + g_2 + \dots + g_d$, where g_i is homogeneous of degree i . A line through p has the form $\{t(a_1, \dots, a_n) \mid t \in k\}$ for a uniquely determined $[a_1 : \dots : a_n] \in \mathbb{P}^{n-1}$. Expanding g along such a line, we obtain $g(t(a_1, \dots, a_n)) = tg_1(a_1, \dots, a_n) + t^2g_2(a_1, \dots, a_n) + \dots + t^dg_d(a_1, \dots, a_n)$. The given condition now implies that g_1, \dots, g_{d-1} all vanish on a dense subset L of \mathbb{P}^{n-1} . Then $\overline{L} \subset V(g_i)$ for each $i = 1, \dots, d-1$, hence $g_i = 0$. \square

Let $p \in V(g)$ be a smooth point on a hypersurface $V(g)$ in \mathbb{A}_k^n . Change coordinates so that $p = (0, \dots, 0)$ and expand g near p as $g = g_1 + g_2 + \dots$, where g_i is homogeneous of degree i (note that $g_1 \neq 0$). We say that p is a flexy point if $g_1|g_2$. This is a closed condition, so the subset of $V(g)_{\text{smooth}}$ consisting of its flexy points is either a proper closed subset of $V(g)_{\text{smooth}}$, or all of $V(g)_{\text{smooth}}$ (the latter case can happen only when k has positive characteristic; then $V(g)$ is called a flexy hypersurface).

An irreducible plane curve $C \subset \mathbb{P}_k^2$ is called “funny” if there is a point $p_0 \in \mathbb{P}^2$ such that for all points $p \in C$, the line joining p and p_0 is tangent to C at p . If C is irreducible and is non-funny, then for any point p_0 , there are only finitely many points $p \in C$ for which the line joining p and p_0 is tangent to C at p .

Lemma 10. *Let $C = V(G) \subset \mathbb{P}^2 \simeq V(x_0)$ be an irreducible curve, and let $X = V(G) \subset \mathbb{P}^3$ be the cone over it, with vertex $v = [1 : 0 : 0 : 0]$. Let p be a smooth non-flexy point on C . Then the line joining p and v is the only line $l \subset \mathbb{P}^3$ passing through p which satisfies $I_p(l, X) \geq 3$.*

Proof. Choose coordinates so that $p = [0 : 0 : 0 : 1]$ and so that the tangent line at p to C in \mathbb{P}^2 (or in $\mathbb{A}_{x,y}^2$) is given by $x = 0$. Let g be the dehomogenization of G with respect to z . So, g has the form $g = x + \alpha x^2 + \beta xy + \gamma y^2 + \text{h.o.t.}$ and by assumption, $\gamma \neq 0$. Given now a line l in $\mathbb{A}_{x_0,x,y}^3$ with $p \in l$, described by $x_0 = ta, x = tb, y = tc$ (for $t \in k$), then we have $I_p(l, V(g)) \geq 3$ precisely when $b = c = 0, a \neq 0$. \square

Remark 11. Let $p \in V(g)$ be a non-flexy smooth point on a hypersurface $V(g)$ in \mathbb{A}_k^3 of degree d . Then the number of lines $l \subset \mathbb{A}_k^3$ passing through p with the property that $I_p(l, V(g)) \geq 3$ is either one or two.

Proof of Proposition 3. We know that $\dim C = 1$.

Let $M = 2^{\{R_1, \dots, R_s\}}$ be the collection of all nonempty subsets of $\{R_1, \dots, R_s\}$. For each $x \in V(x_0) - C$, choose a Kakeya line l_x through x whose affine part is contained in E . By the pigeonhole principle applied to the assignment $V(x_0) - C \rightarrow M, x \mapsto l_x \cap V(G)$, there exists a nonempty subset $S \subset \{R_1, \dots, R_s\}$ which is the image of infinitely many points $x \in V(x_0) - C$. The set S must be singleton, say $S = \{R_1\}$. Let $R_1, \dots, R_{s'}$ be the points among R_1, \dots, R_s with the property that for each $i = 1, \dots, s'$, there are infinitely many points $x \in V(x_0) - C$ such that $l_x \cap V(G) = \{R_i\}$. For $i = 1, \dots, s'$, let Ω_i be the set of all $x \in V(x_0) - C$ such that $l_x \cap V(G) = \{R_i\}$. Then the complement of $\cup_{i=1}^{s'} \Omega_i$ in $V(x_0) - C$ is finite. Thus, $\cup_{i=1}^{s'} \Omega_i$

is dense in $V(x_0) - C$, so for some $i \in \{1, \dots, s'\}$, the set Ω_i is dense in $V(x_0) - C$, hence in $V(x_0)$. Say this holds for $i = 1$, and let $R = R_1$. Then $I_R(l_x, V(G)) = d$ for $x \in \Omega_1$ by Bezout's theorem. By Lemma 9, R is a point of $V(G)$ of multiplicity d .

For any $x \in C$, the line joining x and R intersects $V(G)$ at R with intersection multiplicity at least d , and also intersects $V(G)$ at x , hence Bezout's theorem implies that this line is contained in $V(G)$. Therefore, the entire cone over C with vertex R is contained in $V(G)$. Since $V(G)$ is irreducible, this implies that $V(G)$ is precisely the cone over C with vertex R . Choose coordinates so that $R = [1 : 0 : 0 : 0]$; then G does not involve the variable x_0 . Note that C is irreducible (if C were reducible, so would be the cone over it), and $\deg C = \deg V(G) = d$.

Next, for any $x \in C$, there exists a Kekeya line l_x passing through x , which intersects $V(G)$ only at points among $\{R_2, \dots, R_s\}$. Since C is non-flexy curve, the set U consisting of all (smooth) non-flexy points of C is an open dense subset of C . For any $x \in U$ and any line l through x other than the one joining x and R , we know that $I_x(l, V(G)) \leq 2$ by Lemma 10; in particular, since $d \geq 3$, the Kekeya line l_x through x must intersect $V(G)$ again.

Repeat the earlier argument, now for the assignment $U \rightarrow 2^{\{R_2, \dots, R_s\}}, x \mapsto l_x \cap V(G)$ to find a point in $\{R_2, \dots, R_s\}$, say R_2 , and a dense subset $\Omega \subset U$, such that for all $x \in \Omega$, we have $l_x \cap V(G) = \{x, R_2\}$. Let $\overline{R_2} \in C$ be the point of intersection of the line joining R and R_2 with $V(x_0)$. Since C is non-funny, there are at most finitely many smooth points $p \in C$ such that the tangent line to C at p passes through $\overline{R_2}$. Shrinking U if necessary, we can assume that for any $x \in U$, the tangent line to C at x in $V(x_0)$ does not pass through $\overline{R_2}$, and for all $x \in \Omega \subset U$, we have $l_x \cap V(G) = \{x, R_2\}$. For $x \in U$, note that the line joining x and R_2 is not contained in the tangent plane to $V(G)$ at x .

For any $x \in \Omega \subset U$, the Kekeya line l_x is the line joining x and R_2 , so $I_x(l_x, V(G)) = 1$. By Bezout's theorem, $I_{R_2}(l_x, V(G)) = d - 1$, for any $x \in \Omega$.

We claim that R_2 is a point on $V(G)$ of multiplicity $d - 1$. Say $R_2 = [1 : a_1 : a_2 : a_3]$ and set $a = (a_1, a_2, a_3) \in \mathbb{A}_k^3$ (these are fixed once and for all). Note that the dehomogenization of G with respect to the first variable x_0 is just G itself. So, to determine the multiplicity of R_2 on $V(G)$, we have to examine

$$G(a + (x_1, x_2, x_3)) = g_1(x_1, x_2, x_3) + \dots + g_d(x_1, x_2, x_3),$$

where $g_i \in k[x_1, x_2, x_3]$ is homogeneous of degree i . For any $[0 : v_1 : v_2 : v_3] \in \Omega$, examine the intersection multiplicity at a of $V(G)$ and the line $l_{a,v}$ passing through a in the direction $v = [v_1 : v_2 : v_3]$. Note that

$$G(a + tv) = tg_1(v_1, v_2, v_3) + t^2g_2(v_1, v_2, v_3) + \dots + t^dg_d(v_1, v_2, v_3).$$

The condition that $I_a(V(G), l_{a,v}) = d - 1$ means that $g_i(v_1, v_2, v_3) = 0$ for $i = 1, \dots, d - 2$. Thus, for $i = 1, \dots, d - 2$, we have $\Omega \subset V(g_i)$ and since Ω is dense in C , we deduce $C \subset V(g_i)$ for all $i = 1, \dots, d - 2$. However, C has degree d while g_i has degree at most $d - 2$. Therefore, $g_i(x_1, x_2, x_3) = 0$ in $k[x_1, x_2, x_3]$ for $i = 1, \dots, d - 2$.

Since R_2 is a point on $V(G)$ of multiplicity $d - 1$, so is $\overline{R_2}$ on C . If $q \neq \overline{R_2}$ is a non-smooth point of C , the line joining $\overline{R_2}$ and q would have to be contained in C , by Bezout's theorem,

and so by irreducibility, C would be a line (but we are in the case $d \geq 3$). Therefore, indeed, $\overline{R_2}$ is the unique singular point of C . \square

Remark 12. Note that this proof implies that under the assumptions of Proposition 3, one must have $s \geq 2$. So, the construction in the proof of Proposition 2 is optimal in terms of the number of points R_i that one needs to add to $\{g \neq 0\}$ to obtain a Kakeya set.

3 Kakeya subvarieties of affine or projective space

Let $X \subset \mathbb{A}_{x_1, \dots, x_N}^N$ be an n -dimensional subvariety and let $\overline{X} \subset \mathbb{P}_{[x_0: \dots: x_N]}^N$ be its Zariski closure. Define $\Delta = \Delta(X)$ as the set of all directions of lines contained in X :

$$\Delta(X) = \{v \in V(x_0) \mid \text{there exists a line } l \subset X \text{ such that } \bar{l} \cap V(x_0) = \{v\}\}.$$

This is a constructible subset of $V(x_0)$; it is the image under the first projection of $\{(v, l) \mid v \in l\} \subset V(x_0) \times F_1(X)$, where we set $F_1(X) := (F_1(\overline{X}) - F_1(\overline{X} \cap V(x_0)))$; as usual, F_1 stands for the Fano variety of a projective variety.

Proposition 13. *Notation as above, we have $\dim \Delta \leq n - 1$.*

Proof. For any $v \in \Delta$, there is a line $l \subset X$ whose closure contains v . But then $v \in \bar{l} \subset \overline{X}$ and hence $\overline{\Delta} \subset \overline{X}$. If $\dim \Delta \geq n$, then Δ would have to be an irreducible component of \overline{X} , which is impossible since $\Delta \subset V(x_0)$ and $X \subset \mathbb{A}^N$. \square

Note that this proof is again a geometric version of Dvir's argument.

Definition 14. *Let $X \subset \mathbb{A}^N$ be an n -dimensional subvariety. We say that it is a Kakeya subvariety if the inequality $\dim \Delta \leq n - 1$ is an equality.*

3.1 Examples coming from hypersurfaces

Here we give as examples a class of Kakeya varieties.

Proposition 15. *Let $1 \leq d \leq N - 1$ and let $X = V(f) \subset \mathbb{A}^N$ be a hypersurface of degree d . Then X is a Kakeya subvariety of \mathbb{A}^N .*

Proof. Let S_d be the k -vector space of all polynomials in $k[x_1, \dots, x_N]$ of degree at most d , and let $\mathbb{A}(S_d)$ be the affine space associated to S_d . Consider the incidence correspondence

$$\mathcal{A} = \{(X, v) \in \mathbb{A}(S_d) \times V(x_0) \mid \text{some line } l \subset X \text{ has direction } v\} \subset \mathbb{A}(S_d) \times V(x_0).$$

For $v \in V(x_0)$, let Σ_v be the fiber over v under the second projection. Note that $\dim \Sigma_v$ does not depend on the specific $v \in V(x_0)$.

Let \mathbb{G}_v° be the subset of the Grassmanian $\mathbb{G}(1, N)$ consisting of lines through v and not contained in $V(x_0)$. Consider the incidence correspondence

$$\mathcal{B} = \{(X, l) \in \mathbb{A}(S_d) \times \mathbb{G}_v^\circ \mid l \subset \overline{X}\} \subset \mathbb{A}(S_d) \times \mathbb{G}_v^\circ.$$

Under the surjection $\mathcal{B} \rightarrow \mathbb{G}_v^\circ$, each fiber is irreducible of dimension equal to $\dim S_d - d - 1$ (see Lemma 16 below), hence \mathcal{B} is irreducible, of dimension equal to $\dim S_d - d + N - 2$. On the other hand, the image of \mathcal{B} under the first projection is precisely Σ_v (in particular, Σ_v is irreducible). Let $t := \dim \mathcal{B} - \dim \Sigma_v$ (note that t is independent of v), so $\dim \Sigma_v = \dim S_d - d + N - 2 - t$. Therefore, $\dim \mathcal{A} = \dim S_d - d + 2N - 3 - t$.

Consider now the map $\phi : \mathcal{A} \rightarrow \mathbb{A}(S_d)$. The fiber of ϕ over $f \in \mathbb{A}(S_d)$ is the direction set $\Delta(V(f))$, so for each $f \neq 0$ in $\text{Image}(\phi)$ (such f 's certainly exist), we have the chain of inequalities

$$\dim \mathcal{A} - \dim S_d \leq \dim \mathcal{A} - \dim(\text{Image}(\phi)) \leq \dim \phi^{-1}(f) = \dim(\Delta(V(f))) \leq N - 2,$$

hence $t \geq N - d - 1$. On the other hand, we have $t \leq N - d - 1$ by Lemma 17 below, and therefore, all inequalities in the above chain must be equalities. In particular, for any $f \in \text{Image}(\phi) - \{0\}$, we have $\dim \Delta(V(f)) = N - 2$. Moreover, $\dim(\text{Image}(\phi)) = \dim \mathbb{A}(S_d)$ and since the map ϕ is proper (note that $V(x_0)$ is proper over the point, hence so is the basechange map $\mathbb{A}(S_d) \times V(x_0) \rightarrow \mathbb{A}(S_d)$), it is actually surjective. Therefore, in fact, any $f \in \mathbb{A}(S_d) - \{0\}$ belongs to $\text{Image}(\phi)$, and hence the above chain of inequalities holds for any $f \neq 0$. \square

Lemma 16. *Let $v \in V(x_0)$ and let $l \in \mathbb{G}_v^\circ$. Then $\{X \in \mathbb{A}(S_d) \mid l \subset \overline{X}\}$ is an affine space of dimension $\dim S_d - d - 1$.*

Proof. Change coordinates so that $v = [0 : 1 : 0 : \dots : 0]$ and $l = V(x_2, \dots, x_N)$. The set under investigation consists of all polynomials in the ideal (x_2, \dots, x_N) whose degree is at most d . The dimension count follows from inspecting the exact sequence

$$0 \rightarrow (x_2, \dots, x_N)_{\leq d} \rightarrow k[x_1, \dots, x_N]_{\leq d} \rightarrow k[x_1]_{\leq d} \rightarrow 0$$

of k -vector spaces. \square

Lemma 17. *Let $v \in V(x_0)$ and let $1 \leq d \leq N - 1$. There exists a hypersurface $V(f) \subset \mathbb{A}^N$ of degree d in Σ_v such that*

$$\dim\{l \in \mathbb{G}_v^\circ \mid l - \{v\} \subset V(f)\} = N - d - 1.$$

Proof. Without loss of generality, $v = [0 : 1 : 0 : \dots : 0]$. Identify $\mathbb{G}_v^\circ \simeq V(x_1) \cap D_+(x_0) \simeq V(x_1) \subset \mathbb{A}_{x_1, x_2, \dots, x_N}^N$ (slightly abusing notation), via $l \mapsto l \cap V(x_1)$. Consider hypersurfaces $V(f)$ which contain the line $V(x_2, \dots, x_N)$ joining $(0, \dots, 0)$ and v . Any such f can be written as

$$f = g_1 x_1^{d-1} + g_2 x_1^{d-2} + \dots + g_{d-1} x_1 + g_d,$$

where $g_i \in k[x_2, \dots, x_N]$, $\deg(g_i) \leq i$ for each i (with equality holding for some i), and each $g_i \in (x_2, \dots, x_N)$. Then the line joining v and $(0, a_2, \dots, a_N)$ is contained in $V(f)$ if and only if $(a_2, \dots, a_N) \in V(g_1, \dots, g_d) \subset \mathbb{A}_{x_2, \dots, x_N}^{N-1}$. We can certainly pick g_1, \dots, g_d such that $V(g_1, \dots, g_d)$ has dimension $N - 1 - d$; for example, take $g_i = x_{i+1}$ for $i = 1, \dots, d$. \square

3.2 Linear projections yield covers

Next, we investigate the image of a Kekeya subvariety under a linear projection and prove Proposition 5. Note that if $X = \cup X_i$ is the decomposition of X into irreducible components, then $\Delta(X) = \cup \Delta(X_i)$, so in the investigation of $\Delta(X)$, when convenient, we can assume that X is irreducible.

Let $X \subset \mathbb{A}_{x_1, \dots, x_N}^N$ be an irreducible n -dimensional Kekeya subvariety of \mathbb{A}^N , and let d be the degree of its closure $\overline{X} \subset \mathbb{P}_{[x_0, \dots, x_N]}^N$ in \mathbb{P}^N . Let $\Delta \subset V(x_0)$ be the set of directions of lines contained in X . Let $P \in V(x_0) - \overline{X}$ be arbitrary, and let $H \subset \mathbb{P}^N$ be any hyperplane such that $P \notin H$. Let $\pi : \mathbb{P}^N - \{P\} \rightarrow H$ be the linear projection from P to H .

Let $Y = \pi(X) \subset H \cap D_+(x_0)$ and note that $\overline{Y} = \pi(\overline{X})$. By slight abuse of notation, the restriction $\pi : \overline{X} \rightarrow \overline{Y}$ is also denoted by π . Note that π is a finite map; in particular, $\dim Y = n$. When $n < N - 1$, we know that π is birational and \overline{Y} has degree d in H ; when $n = N - 1$, we know that $\overline{Y} = H$ and π has degree d .

Since π is a linear projection, it induces $F_1(\overline{X}) \rightarrow F_1(\overline{Y})$ and $F_1(X) \rightarrow F_1(Y)$. Let $\Delta' \subset H \cap V(x_0)$ be the set of directions of lines contained in Y . Thus, π induces also $\pi : \Delta \rightarrow \Delta'$, and all fibers of this map are finite. Therefore, $\dim \Delta' \geq \dim \pi(\Delta) = \dim \Delta = n - 1$ and hence $Y = \pi(X)$ is a Kekeya subvariety of $\mathbb{A}^{N-1} = H \cap D_+(x_0)$.

We can repeat the process described above and decrease the codimension of X in \mathbb{A}^N . When we get to $N = n + 1$, linear projection π as above will yield a finite map $\overline{X} \rightarrow \mathbb{P}^n = H$ with the property that $\pi(\Delta) \subset V(x_0) \cap H$ is $(n - 1)$ -dimensional. So, $\pi(\Delta)$ will contain an open subset $U \subset V(x_0) \cap H$ and hence for every $v \in U$, there exists a line $l \subset X$ whose projection passes through v .

3.3 A three-dimensional example coming from the Grassmanian $\mathbb{G}(1, 4)$

Consider the 6-dimensional Grassmanian $X = \mathbb{G}(1, 4) \subset \mathbb{P}(\wedge^2 k^5) \simeq \mathbb{P}_{[p_{ij}]}^9$, embedded in the projective space \mathbb{P}^9 as a degree-5 subvariety via Plucker coordinates.

It is known (for example, see [3]) that $\dim F_1(X) = 8$. So, if $W \subset \mathbb{P}^9$ is a 6-dimensional linear subspace, the expected dimension of $F_1(X \cap W)$ is 2. By the main result in [3], this is one of the examples of 3-dimensional projective varieties whose Fano variety is 2-dimensional.

Consider $W = V(p_{12} - p_{15}, p_{23} - p_{25}, p_{34})$. Then $X' = X \cap W$ is irreducible and of dimension 3. We perform 3 appropriate linear projections, whose composition, in affine coordinates, is given as follows:

$$X' = V(z - xy, ay - b + a, -bx + c) \hookrightarrow \mathbb{A}^6 \rightarrow \mathbb{A}^3, \quad (a, b, c, x, y, z) \mapsto (a - x + y, b - z, c).$$

Consider now any direction $[\alpha : 1 : \gamma]$, with $\alpha \neq 0$. Then $a = \alpha^2 t, b = \alpha t, c = \alpha \gamma t, x = \gamma, y = \frac{1}{\alpha} - 1, z = \gamma(\frac{1}{\alpha} - 1)$ defines a line in \mathbb{A}^6 which is contained in X' , and whose image under $X' \rightarrow \mathbb{A}^3$ is a line in direction $[\alpha^2 : \alpha : \alpha \gamma] = [\alpha : 1 : \gamma]$. In other words, the map $X' \rightarrow \mathbb{A}^3$ is Kekeya cover in the sense of Definition 4.

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